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Decorated Ising model in a magnetic field

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Abstract. A study of the two-dimensional Ising model on a rectangular lattice decorated with ν-dimensional classical vector spins or Ising spins of magnitude s in an external field is presented. Exact expressions are obtained for the magnetisation and correlation functions, and the phase diagram is also determined exactly. It is shown that when there is no direct interaction between the spins of the host lattice there is a critical field h_c, above which the system does not present long range order, and that this h_c does not depend on ν or s. The isothermal susceptibility related to the decorating spins is also obtained exactly, and it is shown that it diverges logarithmically at the transition temperature only for non-vanishing external field. A discussion is presented on the solution of equivalent models, and as a by-product it is shown how to map the model onto the two-dimensional eight-vertex model with temperature dependent parameters in an arbitrary staggered field.

1. Introduction

The exact knowledge of the effect of an external magnetic field on the critical behaviour of the Ising model is still restricted to one-dimensional lattices. Although many years have passed since the zero-field solution was obtained for the two-dimensional square lattice (Onsager 1944) no exact result has been obtained so far for the problem in the presence of an external field.

There are, however, some decorated Ising models in a magnetic field which are amenable to exact solutions. The first of these models was introduced by Fisher (1959a) and its solution also obtained in detail by Fisher (1960a, b). It corresponds to the case when the decorating spin is an Ising spin of magnitude $\frac{1}{2}$.

The purpose of this paper is to study exactly the effect of the external field on the two-dimensional decorated Ising model when the decorating spins are ν-dimensional classical spins or Ising spins of magnitude s. The study corresponds to an extension of Fisher's model (Fisher 1960a, b), and the critical behaviour of the system is analysed under more general circumstances. We also discuss other models which present the same solution and the relation of one of those to the eight-vertex model.

In § 2 we present the basic general results for the decorated model in an arbitrary external field. In § 3 we discuss the exactly soluble cases and a mapping onto the eight-vertex model. In the final section specific cases are presented and the main results discussed.

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2. General results

The model that we are going to consider is the whole decorated Ising model on a rectangular lattice whose Hamiltonian is given by

$$H = -\sum_{(ij)} J\sigma_{ij}\sigma_{ij+1} - \sum_{(ij)} \bar{J}\sigma_{ij}\sigma_{i+1j} - \sum_{(ij)} \tau_{ij,ij+1}(J_1\sigma_{ij} + J_2\sigma_{ij+1})$$
$$- \sum_{(1j)} \tau_{ij,i+1j}(\bar{J}_1\sigma_{ij} + \bar{J}_2\sigma_{i+1j}) - \sum_{(ij)} \bar{h}\sigma_{ij} - \sum_{(ij)} h(\tau_{ij,i+1j} + \tau_{ij,ij+1}) \tag{1}$$

where $\sigma = \pm 1$ and τ denotes an Ising spin of magnitude s or the νth component of the classical ν-dimensional classical spin. The partition function is then given by

$$Z = \sum_{\{\tau\}} \sum_{\{\sigma\}} \exp(-\beta H) \tag{2}$$

where as usual $\beta = 1/k_B T$, and the sum over the set $\{\tau\}$ means integrals on the hypersolid angles associated with each classical vector spin (Gonçalves 1982), when τ is the νth component of the classical vector spin. The sum on $\{\tau\}$ can be performed immediately (Gonçalves 1982, Goncalves and Horiguchi 1984) and we can write

$$Z = (r\bar{r})^N \sum_{\{\sigma\}} \exp(-\beta H') \tag{3}$$

where N is the number of sites of the lattice,

$$r = [F(J_1 + J_2 - h)F(J_1 + J_2 + h)F(J_1 - J_2 - h)F(J_1 - J_2 - h)]^{1/4} \tag{4}$$

and H' is given by

$$H' = -\sum_{(ij)} (J + J')\sigma_{ij}\sigma_{ij+1} - \sum_{(ij)} (\bar{J} + \bar{J}')\sigma_{ij}\sigma_{ij+1} - \sum_{(ij)} (\bar{h} + \tfrac{1}{2}(h_1' + \bar{h}_1' + h_2' + \bar{h}_2'))\sigma_{ij} \tag{5}$$

with

$$J' = \frac{k_B T}{4} \ln\left(\frac{F(J_1 + J_2 - h)F(J_1 + J_2 + h)}{F(J_1 - J_2 - h)F(J_1 - J_2 + h)}\right)$$
$$h_1' = \frac{k_B T}{2} \ln\left(\frac{F(J_1 + J_2 + h)F(J_1 - J_2 + h)}{F(J_1 + J_2 - h)F(J_1 - J_2 - h)}\right) \tag{6}$$
$$h_2' = \frac{k_B T}{2} \ln\left(\frac{F(J_1 + J_2 + h)F(J_1 - J_2 - h)}{F(J_1 + J_2 - h)F(J_1 - J_2 + h)}\right)$$

and $\bar{r}$, $\bar{J}'$, $\bar{h}_1'$, $\bar{h}_2'$ are obtained from equations (4) and (6) by substituting J_1 and J_2 by $\bar{J}_1$ and $\bar{J}_2$ respectively. The function $F(z)$ which appears in the previous equation is given by (Gonçalves and Horiguchi 1984)

$$F(z) = \frac{\sinh[\beta z(s + \frac{1}{2})]}{\sinh(\beta z/2)} \tag{7}$$

when τ is an Ising spin of magnitude s, and by

$$F(z) = (2\pi)^{\nu/2}(\beta z)^{1-\nu/2} I_{\nu/2-1}(\beta z) \tag{8}$$

where $I_{\nu/2-1}$ is the modified Bessel function of first kind, when τ is the νth component of the classical vector spin.

An exact solution for the problem is only possible when the effective field is equal to zero. This defines a surface in the parameter space, whose equation is given by

$$\bar{h}+\tfrac{1}{2}(h_1'+\bar{h}_1'+h_2'+\bar{h}_2')=0 \tag{9}$$

where we can get a solution for a non-zero external field. Although a general analysis of the problem can be carried out, in this paper we will be interested only in the case where $\bar{h}$ is equal to zero. This corresponds to having a non-uniform field applied to the lattice (Fisher 1959a).

The thermodynamic properties, provided equation (9) is satisfied, can easily be found from the known results for the two-dimensional Ising model (see, for instance, McCoy and Wu 1973).

The magnetisation and spin correlations are obtained by using a procedure adopted previously (Gonçalves and Horiguchi 1984), namely to write the identity

$$\begin{aligned}\sum_{\tau_{12}}\{\exp[\beta(J_1\sigma_1+J_2\sigma_2)\tau_{12}+\beta h\tau_{12}]\tau_{12}\}\\ &= G(J_1\sigma_1+J_2\sigma_2+h)\\ &\equiv F(J_1\sigma_1+J_2\sigma_2+h)(x_1\sigma_1\sigma_2+x_2\sigma_1+x_3\sigma_2+x_4)\end{aligned} \tag{10}$$

where G is given for each kind of decorating spin respectively by (Gonçalves and Horiguchi 1984)

$$G(z)=sF(z)B_s(sz) \tag{11}$$

and

$$G(z)=(2\pi)^{\nu/2}(\beta z)^{1-\nu/2}I_{\nu/2}(\beta z) \tag{12}$$

where $B_s(z)$ is the Brillouin function. The x given by equation (10) are easily obtained and are given by

$$\begin{aligned}x_1&=\tfrac{1}{4}[\bar{F}(J_1+J_2+h)+\bar{F}(J_1-J_2-h)-\bar{F}(J_1+J_2-h)-\bar{F}(J_1-J_2+h)]\\ x_2&=\tfrac{1}{4}[\bar{F}(J_1+J_2+h)+\bar{F}(J_1+J_2-h)+\bar{F}(J_1-J_2+h)+\bar{F}(J_1-J_2-h)]\\ x_3&=\tfrac{1}{4}[\bar{F}(J_1+J_2+h)+\bar{F}(J_1+J_2-h)-\bar{F}(J_1-J_2+h)-\bar{F}(J_1-J_2-h)]\\ x_4&=\tfrac{1}{4}[\bar{F}(J_1+J_2+h)+\bar{F}(J_1-J_2+h)-\bar{F}(J_1+J_2-h)-\bar{F}(J_1-J_2-h)]\end{aligned} \tag{13}$$

where $\bar{F}(z)$ is defined as

$$\bar{F}(z)=G(z)/F(z). \tag{14}$$

There is also another set of $\{x\}$, which we will call $\{\bar{x}\}$, which is obtained from equation (13) by substituting J_1 and J_2 by $\bar{J}_1$ and $\bar{J}_2$ respectively.

The magnetisation per vertex is determined immediately by using equations (10) and (13), and can be expressed as

$$M=(1+x_2+x_3+\bar{x}_2+\bar{x}_3)\langle\sigma\rangle+x_1\langle\sigma_1\sigma_2\rangle_{\rm h}+\bar{x}_1\langle\sigma_1\sigma_2\rangle_{\rm v}+x_4+\bar{x}_4 \tag{15}$$

where $\langle\sigma\rangle$ is the magnetisation of the effective Ising model described by H', and $\langle\sigma_1\sigma_2\rangle_{\rm h}$ and $\langle\sigma_1\sigma_2\rangle_{\rm v}$ are the nearest-neighbour pair correlation functions along the horizontal and vertical directions respectively. The spontaneous magnetisation is obtained from equation (15) by taking the limit h, $\bar{h} \to 0$ and this implies

$$x_1 = x_4 = 0 \qquad \bar{x}_1 = \bar{x}_4 = 0 \tag{16}$$

which gives

$$M = (1 + x_2 + x_3 + \bar{x}_2 + \bar{x}_3)\langle \sigma \rangle. \tag{17}$$

By using equation (10), the pair spin correlation functions for the decorating spins are also easily expressed in terms of correlation functions in the effective Ising model, and the various pair spin correlations are given by

$$\begin{aligned}
\langle \tau_{ij,ij+1} \tau_{i'j',i'j'+1} \rangle &= \frac{\Sigma_{\{\sigma\}} \exp(-\beta H') c_{ij} c_{i'j'}}{\Sigma_{\{\sigma\}} \exp(-\beta H')} \\
\langle \tau_{ij,ij+1} \tau_{i'j',i'+1j'} \rangle &= \frac{\Sigma_{\{\sigma\}} \exp(-\beta H') c_{ij} \bar{c}_{i'j'}}{\Sigma_{\{\sigma\}} \exp(-\beta H')} \\
\langle \tau_{ij,i+1j} \tau_{i'j',i'+1j'} \rangle &= \frac{\Sigma_{\{\sigma\}} \exp(-\beta H') \bar{c}_{ij} \bar{c}_{i'j'}}{\Sigma_{\{\sigma\}} \exp(-\beta H')}
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
c_{ij} &= x_1 \sigma_{ij} \sigma_{ij+1} + x_2 \sigma_{ij} + x_3 \sigma_{ij+1} + x_4 \\
\bar{c}_{ij} &= \bar{x}_1 \sigma_{ij} \sigma_{i+1j} + \bar{x}_2 \sigma_{ij} + \bar{x}_3 \sigma_{i+1j} + \bar{x}_4.
\end{aligned} \tag{19}$$

The susceptibility associated with the decorating spins is naturally given by

$$\chi = \frac{\partial M_D}{\partial h} = \frac{1}{\beta N} \frac{\partial^2 (\ln Z)}{\partial h^2} \tag{20}$$

which is immediately evaluated provided the partition function is known as a function of h.

3. Exactly soluble cases

For the special case where $\bar{h} = 0$ equation (9) is trivially satisfied provided we have the relations

$$\bar{J}_1 = -J_1 \qquad \bar{J}_2 = -J_2. \tag{21}$$

For $J = \bar{J}$, $J_1 = J_2$ and $s = \frac{1}{2}$ or $\nu = 1$, the solution reduces to the one obtained by Fisher (1960a, b). By using the condition shown in equation (21) and bearing in mind that $\bar{F}(z)$ is an odd function, we can write the total magnetisation in the form

$$M = \langle \sigma \rangle + 2x_1 \langle \sigma_1 \sigma_2 \rangle + 2x_4 \tag{22}$$

where we have also assumed $J = \bar{J}$. It should be noted that this condition will also be assumed in all subsequent discussions. By comparing equations (17) and (22) we conclude immediately that the terms in x_1 and x_4 correspond to the induced magnetisation. It is worth noting that $x_2 = -\bar{x}_2$ and $x_3 = -\bar{x}_3$ when the exchange constants satisfy equation (21), even for h different from zero. Therefore the magnetisation associated with the long range order, even in the presence of the field h, is the magnetisation of the host lattice, namely $\langle\sigma\rangle$. In passing we would like to mention that we could define a long range magnetisation as

$$M=(1+x_2+x_3-\bar{x}_2-\bar{x}_3)\langle\sigma\rangle+(x_1-\bar{x}_1)\langle\sigma_1\sigma_2\rangle+x_4-\bar{x}_4 \tag{23}$$

which for the parameters given by equation (21) reduces to

$$M=(1+2x_2+2x_3)\langle\sigma\rangle. \tag{24}$$

For $\bar{h}=0$ the critical behaviour is obtained just by studying $\langle\sigma\rangle$ which is the magnetisation of the host lattice.

The transition temperature is obtained from the Onsager solution (Onsager 1944) and is given by the equation (see, for example, McCoy and Wu 1973)

$$K+K'=\pm\tfrac{1}{2}\ln(1+\sqrt{2}) \tag{25}$$

where $K=\beta J$ and $K'=\beta J'$.

The critical field is determined by considering the solution of equation (25) in the limit $\beta\to\infty$. Without loss of generality we will consider J_1, $J_2>0$ and $J_1>J_2$. It should be noted that this is so because J' satisfies the relations

$$\begin{aligned} &J'(J_1,J_2)=J'(J_2,J_1) \qquad J'(J_1,J_2)=J'(-J_1,-J_2)\\ &J'(J_1,J_2)=-J'(J_1,-J_2). \end{aligned} \tag{26}$$

It should also be noticed that the expressions for the critical field are valid only for $J\leq 0$. For $J>0$ the transition temperature is always finite.

When the decorating spin is an Ising spin of magnitude s we have for the critical field the result

$$h_c=(2J/s)+(J_1+J_2) \qquad \text{for } J_1+J_2>h \text{ and } J_1-J_2<h. \tag{27}$$

For $h>J_1+J_2$ and $J=0$ there is no macroscopic order provided $J_1-J_2<h$.

Identically when the decorating spins are unit ν-dimensional classical spins the critical field is obtained by considering the asymptotic behaviour of the modified Bessel function (Abramowitz and Stegun 1965) and we get a similar result

$$h_c=2J+(J_1+J_2) \qquad \text{for } J_1+J_2>h \text{ and } J_1-J_2<h. \tag{28}$$

If we consider $|\bar{S}|=\nu^{1/2}$ and renormalise the exchange constants as J_1, $J_2\to\nu^{1/2}J_1$, $\nu^{1/2}J_2$ (Goncalves and de Almeida 1983) and the magnetic field $h\to\nu^{1/2}h$ we can write the previous equation in the form

$$h_c=(2J/\nu)+(J_1+J_2) \tag{29}$$

with J_1 and J_2 restricted to the same conditions. In both cases for $J=0$ and $h>J_1+J_2$ there is no macroscopic order provided $J_1-J_2<h$.

The isothermal susceptibility, given by equation (20), can be calculated from the expression

$$\chi=\partial(2x_1\langle\sigma_1\sigma_2\rangle+2x_4)/\partial h \tag{30}$$

where all the terms are known.

The model discussed so far is equivalent to the one where we have uniform J_1 and J_2, and an external field h acting on the spins decorating the horizontal bonds and an external field $-h$ acting on the spins decorating the vertical bonds. It is also equivalent to the model where we have uniform exchange constants J_1 and J_2, in an external staggered field acting on the decorating spins. In this case the susceptibility defined by equation (20) will correspond to the staggered susceptibility. The model can also be exactly solved when $J_1 = J_2$ and $\bar{J}_1 = \bar{J}_2$, and shown under some conditions to be equivalent to the eight-vertex model (see, for instance, Baxter 1982) in a staggered field when $\nu = 1$ or $s = \frac{1}{2}$. This result can easily be obtained by considering the limit $J \to 0$ and decimating the spins of the host lattice. After this procedure the partition function can be written as

$$Z_D = A^{N^2} \sum_{\{\sigma\}} \exp(-\beta H'') \tag{31}$$

where

$$A = [16^2 \cosh^2 2K_1 \cosh^2 2\bar{K}_1 \cosh(2K_1 + 2\bar{K}_2) \cosh(2K_1 - 2\bar{K}_1)]^{1/8} \tag{32}$$

and

$$\begin{aligned} H'' = &-\sum_{(ij)} J_S \sigma_{ij}\sigma_{ij+1}\sigma_{i+1j}\sigma_{i+1j+1} - \sum_{(ij)} J_H \sigma_{ij}\sigma_{i+1j+1} \\ &-\sum_{(ij)} J_V \sigma_{ij+1}\sigma_{i+1j} - \sum_{(ij)} J_P(\sigma_{ij}\sigma_{ij+1} + \sigma_{ij}\sigma_{i+1j}) - \sum_{(ij)} h_s \sigma_{ij} \end{aligned} \tag{33}$$

with

$$\begin{aligned} J_S &= \ln\left(\frac{\cosh(2K_1 + 2\bar{K}_1)\cosh(2K_1 - 2\bar{K}_1)}{\cosh^2 2K_1 \cosh^2 2\bar{K}_1}\right)^{1/8} \\ J_H &= \ln\left(\frac{\cosh^2 2K_1 \cosh(2K_1 + 2\bar{K}_1)\cosh(2K_1 - 2\bar{K}_1)}{\cosh^2 2\bar{K}_1}\right)^{1/8} \\ J_V &= \ln\left(\frac{\cosh^2 2\bar{K}_1 \cosh(2K_1 + 2\bar{K}_1)\cosh(2K_1 - 2\bar{K}_1)}{\cosh^2 2K_1}\right)^{1/8} \\ J_P &= \ln\left(\frac{\cosh(2K_1 + 2\bar{K}_1)\cosh(2K_1 - 2\bar{K}_1)}{\cosh(2K_1 - 2\bar{K}_1)}\right)^{1/8}. \end{aligned} \tag{34}$$

Since the partition function Z_D is known we can write

$$Z \equiv \sum_{\{\sigma\}} \exp(-\beta H'') = Z_D / A^{N^2}. \tag{35}$$

It should be noted that in the previous expressions we have redefined the labels of the decorating spins after the decimation of the spins of the host lattice. The decorating spins naturally form a new rectangular lattice.

Equations (31)–(35) allow the determination of the thermodynamic properties of the eight-vertex model for a dependent set of parameters $\{J_S, J_V, J_H, J_P\}$ and arbitrary staggered field h_s. Although there are some limitations as far as the parameters are concerned we believe that this kind of mapping is worthwhile mentioning as a by-product of our calculation.

4. Results and conclusions

We will proceed to a final analysis of the results obtained so far, near the transition temperature at zero field and for small field. In all the results that we will be looking at we will restrict ourselves to the limiting case, namely $J_1 = J_2$ which contains the main features of the model.

The transition temperature for small field can be obtained from equation (25) by expanding the effective exchange constant

$$K_{\rm eff} = \alpha K_1 + K' \tag{36}$$

where

$$\alpha = J/J_1 \tag{37}$$

in the neighbourhood of the transition temperature at zero field. The expansion will give

$$K_{\rm eff} = K_{\rm eff}^c + D_1(K_1 - K_1^c) + D_2(\beta_c h)^2 + \mathrm{O}[(K_1 - K_1^c)^2] + \mathrm{O}[h^4] \tag{38}$$

where

$$D_1 = \alpha + \frac{F'(2J_1)}{F(2J_1)}\bigg|_{T=T_c}$$

$$D_2 = \frac{1}{4}\left[\frac{F''(2J_1)}{F(2J_1)}\bigg|_{T=T_c} - \frac{F''(0)}{F(0)} - \left(\frac{F'(2J_1)}{F(2J_1)}\bigg|_{T=T_c}\right)^2\right] \tag{39}$$

and where we have used the notation $F^n = \mathrm{d}^n F/\mathrm{d}(\beta z)^n$. By substituting the previous result in equation (25) we get

$$T = T_c + \frac{h^2}{k_B J_1}\frac{D_2}{D_1}. \tag{40}$$

A similar expansion can be obtained for the susceptibility (equation (30)). This is immediately obtained by considering the expansion for $\langle\sigma_1\sigma_2\rangle$ near $K_{\rm eff}$ as (McCoy and Wu 1973)

$$\langle\sigma_1\sigma_2\rangle \sim C_1 + f(K_{\rm eff}^c)(K_{\rm eff} - K_{\rm eff}^c)\ln|K_{\rm eff} - K_{\rm eff}^c|. \tag{41}$$

By using equation (38) the previous equation can be written in the form

$$\langle\sigma_1\sigma_2\rangle \sim C_1 + f(K_{\rm eff}^c)D_1(K_1 - K_1^c)\ln|K_1 - K_1^c| + f(K_{\rm eff}^c)D_2\beta_c^2 h^2 \ln|K_1 - K_1^c|. \tag{42}$$

Finally by expanding x_1 and x_4 and keeping the terms on first order in the magnetic field only, we obtain

$$x_1 \simeq \beta_c D_3 h \qquad x_2 \simeq \beta_c D_4 h \tag{43}$$

where

$$D_3 = \tfrac{1}{2}[\bar F'(2J_1)|_{T=T_c} - \bar F'(0)]$$

$$D_4 = \tfrac{1}{2}[\bar F'(2J_1)|_{T=T_c} + \bar F'(0)]. \tag{44}$$

We can then write from equation (30) the result

$$\chi \sim 2\beta_c[D_4 + D_3C_1 + D_3D_1 f(K_{\rm eff}^c)(K_1 - K_1^c)\ln|K_1 - K_1^c| + 3f(K_{\rm eff}^c)D_3D_2\beta^2h^2\ln|K_1 - K_1^c|] \tag{45}$$

where we have kept the constant and singular terms only.

The behaviours shown in (40), (42) and (45) are independent of s or ν and are identical to the ones obtained by Fisher (1960a) for $s = \frac{1}{2}$ although the constants contained in these expressions depend on s or ν. As we see the zero-field susceptibility is finite at any temperature and diverges logarithmically at T_c for finite field. In particular

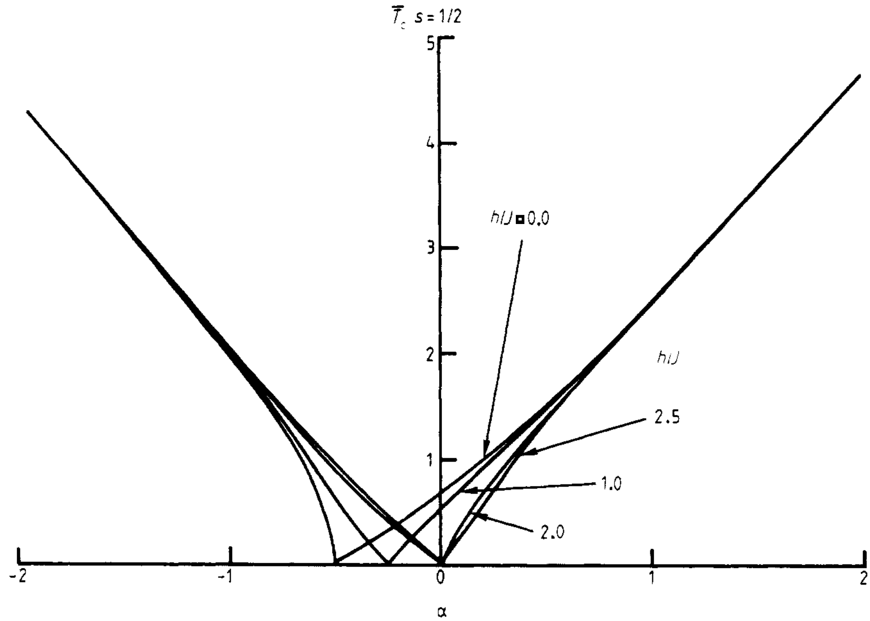

Figure 1. Transition temperature $\bar{T}_c$ ($\bar{T}_c = k_B T_c/J$) as a function of α ($\alpha = J'/J$) for various h and $s = \frac{1}{2}$.

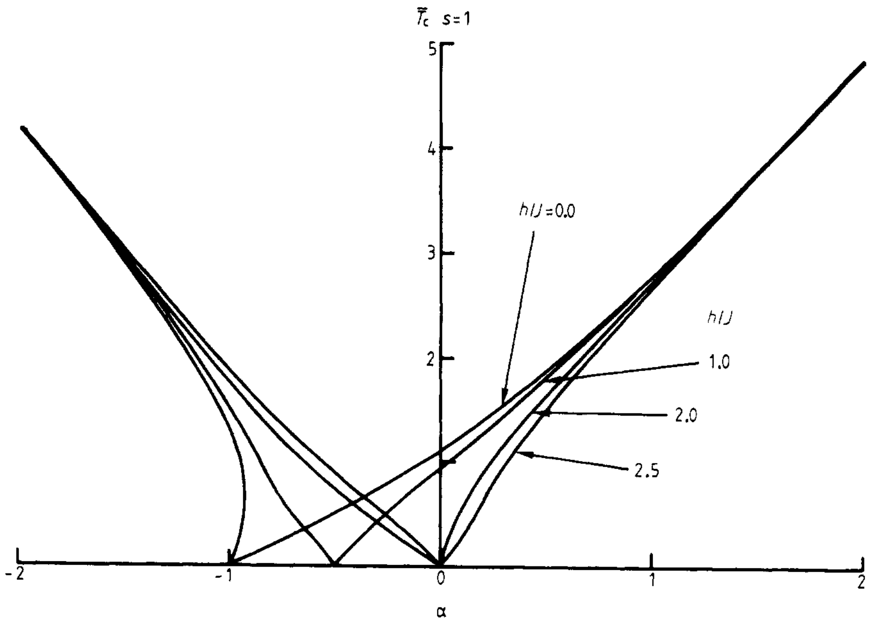

Figure 2. As figure 1 for $s = 1$.

equation (45), in the zero-field limit, is identical to the rigorous result obtained by Fisher (1959b) for the two-dimensional antiferromagnetic Ising model. This result is a special case of the universal behaviour expected for the susceptibiliy, namely

$$\chi \sim A + B|K - K_c|^{1-\alpha} \tag{46}$$

where A and B are constants and α is the specific heat critical exponent.

The phase diagrams are shown in figures 1–3 for $s=\frac{1}{2}$, $s=1$, $\nu=3$ and for various values of the field. For a given field when T_c is equal to zero the spins of the host lattice behave as free spins since at this point J_{eff} is equal to zero. The ordered states located on the right of this point are ferromagnetic and the ones on the left antiferromagnetic. As we see from the figures these regions are functions of the applied field up to the maximum value equal to $2J$. For fields larger than $2J$ the ordered state will be ferromagnetic or antiferromagnetic as J' is positive or negative, respectively.

Another important effect of the field is to suppress the antiferromagnetic state which exists at higher temperatures than the ferromagnetic state, for a given set of parameters. In the absence of the field they do exist for $\nu > 1$ and $s > \frac{1}{2}$ (Horiguchi and Gonçalves 1983), and as we see from figures 2 and 3 they are clearly suppressed by the field.

Finally we would like to mention that the critical exponents are not modified by the field nor the decoration. This means that the model belongs to the same universality class as the two-dimensional Ising model. This result is obtained by following the procedure adopted by Horiguchi (1984) and Gonçalves (1985) in the study of other decorated models. In passing we would like to mention that this is consistent with Fisher's general results (Fisher 1968). Even if we had a constraint in our model there would be no renormalisation of the critical exponents since α is equal to zero for the undecorated lattice.

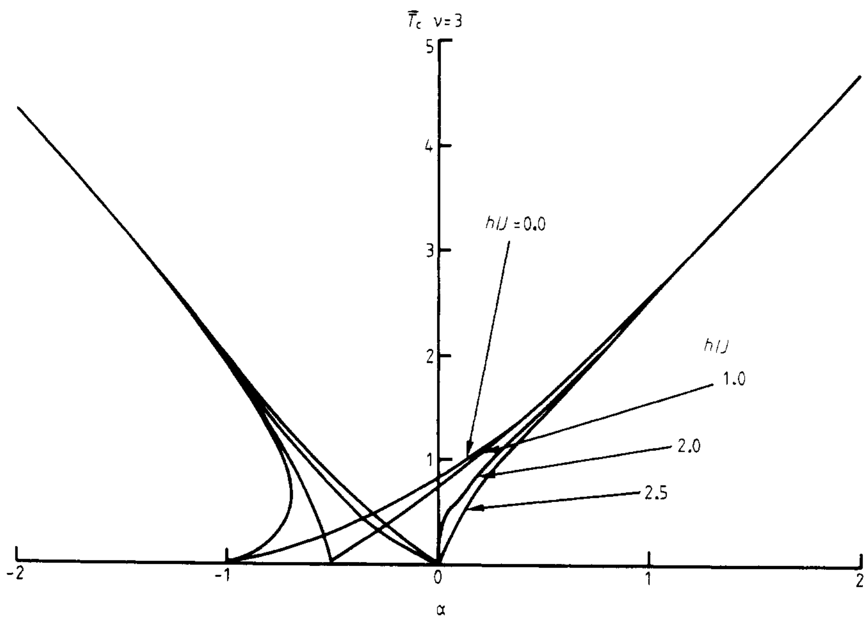

Figure 3. As figure 1 for $\nu = 3$.

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